Math 10A with Professor Stankova
Worksheet, Discussion #20; Wednesday, 10/11/2017
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Substitution Rule

Example

1. Find $\int_0^{16} \sqrt{4-\sqrt{x}}dx$.

> **Solution:** We guess that $u = \sqrt{4-\sqrt{x}}$ and hence $u^2 = 4-\sqrt{x}$ so $\sqrt{x} = 4-u^2$ and $x = (4-u^2)^2 = u^4 - 8u^2 + 16$. Thus, we have that $dx = (4u^3 - 16u)du$. When $x = 0$, then $u = \sqrt{4-0} = 2$ and when $x = 16$, then $u = \sqrt{4-\sqrt{16}} = 0$. Thus
>
> $$\int_0^{16} \sqrt{4-\sqrt{x}}dx = \int_2^0 u(4u^3 - 16u)du = \frac{4}{5}u^5 - \frac{16}{3}u^3\Big|_2^0 = (0-0)-(4/5\cdot 2^5 - 16/3\cdot 2^3)$$
>
> $$= 0 - (-128/5 - 128/3) = \frac{256}{15}.$$

2. Find $\int \sin(x)\cos(x)dx$ using $u = \sin(x)$ and then $u = \cos(x)$. Comment on the result.

> **Solution:** If $u = \sin(x)$, we have $du = \cos(x)dx$ and so
>
> $$\int \sin(x)\cos(x)dx = \int udu = \frac{u^2}{2} + C = \frac{\sin^2(x)}{2} + C.$$
>
> If $u = \cos(x)$, then $du = -\sin(x)dx$ and so
>
> $$\int \sin(x)\cos(x)dx = \int -udu = \frac{-u^2}{2} + C = \frac{-\cos^2(x)}{2} + C.$$
>
> These look like different answers but they are actually the same because $\sin^2(x) + \cos^2(x)$ and so the difference in notation is "hidden" in the constant.

Problems

3. Find $\int \frac{\ln x}{x}dx$.

Solution: Let $u = \ln x$, then $du = \frac{dx}{x}$, so we have that

$$\int \frac{\ln x}{x} dx = \int u du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C.$$

4. Find $\displaystyle\int \frac{1}{x \ln x} dx$.

Solution: Let $u = \ln x$, then $du = \frac{1}{x} dx$ so

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln |u| + C = \ln |\ln x| + C.$$

5. Find $\displaystyle\int x\sqrt{1-x} dx$.

Solution: Let $u = 1 - x$ and so $du = -dx$ and $x = 1 - u$ so this is

$$\int x\sqrt{1-x} = \int (1-u)\sqrt{u}(-du) = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2(1-x)^{5/2}}{5} - \frac{2(1-x)^{3/2}}{3} + C.$$

6. Find $\displaystyle\int_0^{\sqrt{\pi}} x \cos(x^2) dx$.

Solution: Let $u = x^2$ and so $du = 2xdx$ and when $x = 0$, then $u = 0$ and when $x = \sqrt{\pi}$, then $u = \pi$ so we have

$$\int_0^{\sqrt{\pi}} x \cos(x^2) dx = \int_0^{\pi} \frac{\cos(u) du}{2} = \frac{\sin u}{2}|_0^{\pi} = 0.$$

7. Find $\displaystyle\int \sin(x) \sec^2(x) dx$.

Solution: We rewrite $\sec^2(x) = \frac{1}{\cos^2(x)}$. Let $u = \cos(x)$ so that $du = -\sin x dx$ and hence

$$\int \sin(x) \sec^2(x) dx = \int -u^{-2} du = \frac{1}{u} + C = \frac{1}{\cos(x)} + C = \sec(x) + C.$$

8. Find $\int 2xe^{e^{x^2}} e^{x^2} dx$.

Solution: We first try $u = x^2$ so $du = 2xdx$ and hence

$$\int 2xe^{e^{x^2}} e^{x^2} dx = \int e^{e^u} e^u du.$$

Now let $v = e^u$ so $dv = e^u du$ and hence

$$= \int e^v dv = e^v + C = e^{e^u} + C = e^{e^{x^2}} + C.$$